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BOUNDARY VALUE PROBLEMS IN NONLOCAL THEORY OF ELASTICITY

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In recent years a large number of papers have been devoted to the development of various models of elastic media with microstructure. An analysis shows that in all these models there is some scale parameter l, which can characterize the discreteness, the long-range effectiveness, the scale of correlation, and so forth. The appropriate theories can be regarded as weakly or strongly nonlocal. The former are represented by the continuum theory of Cosserat, the couple-stress theory, the multipolar theory of elasticity, and so forth (e.g. see [1-4]). All these can be interpreted as the next approximation with respect to the usual (local) theory of elasticity. The parameter l in these cases must be considered as small.

Strongly nonlocal theories which do not assume the smallness of l, were examined in [5-7] (see also review [8]) for unbounded media.

In this paper boundary value problems of nonlocal theory are examined; the transition from exact to approximate models is investigated; a connection is established with boundary value problems of weakly nonlocal theories [9, 10] in the formulation of which the physical significance of boundary conditions was previously unclear.

The major portion of this work is devoted to one-dimensional problems. In Sect. 1, coupling conditions of two media with microstructure are examined, the analog to Green's formula is constructed, the fundamental boundary value problems and their equivalent integral equations are written down. In Sect. 2, the structure of the general solution of equations of motion for a homogeneous medium is examined. It is shown that the problem is reduced to the determination of roots of the energy operator in the complex plane of wave numbers. Green's function is constructed. Characteristic differences between the nonlocal and the classical theory are examined.

Section 3 is devoted to various approximate models and their regions of applicability. The long-wave approximation is compared to the approximation developed in this paper using first roots of the energy operator. The advantages of the latter approximation will be, the correct description of phenomena for which waves with the length of the order l are essential, the preservation of the principal terms of the asymptotics, and the possibility of correct approximate formulation of the boundary value problems. In Sect. 4, as an illustration the exact and approximate solutions of the fundamental problems are examined for the semi-bounded region.

In Sect. 5 some generalizations are presented for the case of a three-dimensional medium with central interaction.

1. The equations of motion of an inhomogeneous, linearly elastic, one-dimensional medium with nonlocal interaction have the form [11]

$$-\omega^2\rho(x)u(x,\omega) + \int\Phi(x, x')u(x',\omega)\,dx' = q(x,\omega) \qquad (1.1)$$

Here u is the displacement, q are external forces, ρ is the density of the mass, ω is the parameter of Fourier transformation with respect to time (or, which is the same, it is the frequency of established harmonic oscillations), $\Phi(x, x')$ is the kernel of the elastic energy operator. This kernel characterizes the nonlocal interaction.

In the following text, the explicit dependence of field quantities on ω will not be indicated. For expressions of the type (1.1), as a rule, the operator notation will be utilized

$$\Phi_\omega u \equiv (-\omega^2\rho + \Phi)u = q$$

The kernel of the operator Φ must satisfy the following conditions

$$\Phi(x, x') = \Phi(x', x), \qquad \int\Phi(x, x')dx' = 0 \qquad (1.2)$$

It follows from here that $\Phi(x, x')$ can be represented in the form

$$\Phi(x, x') = \psi(x)\delta(x - x') - \Psi(x, x'), \qquad \psi(x) = \int\Psi(x, x')dx' \qquad (1.3)$$

Here $\Psi(x, x')$ is the striffness of the elastic coupling (*), which connects points x and x'.

It is subsequently assumed that the long-range action is random, but limited, i.e. there exists a characteristic radius l of long-range forces such that $\Psi(x, x') = 0$ for $|x - x'| > l$.

Equation (1.1) can describe the motion of a continuous medium with long-range interaction and also of a discrete chain with interaction of any number of neighbors. It was shown in [5] that in the latter case the Fourier transforms of field variables (which will be designated by the same letter, but with the argument k) must be concentrated in k space on the section $|k| \leqslant \pi / a$, where a is the distance between nodes of the lattice. In the limiting case of interaction between a large number of neighbors ($l \gg a$) the equations of the chain for not too short waves transform into equations of the continuum with one scale parameter l.

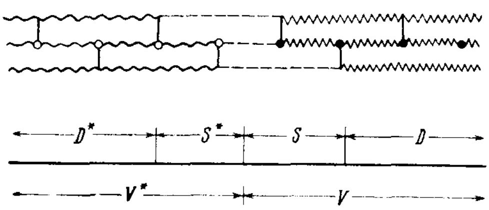

Fig. 1

Let us examine the problem of coupling between two media with different characteristics (the characteristics of the second medium will be designated by an asterisk). In Fig. 1 the coupling of chains with interaction of two neighbors and the diagram for coupling in the general case are shown. In each medium it is possible to select boundary regions S and S^* with a width of the order of l and l^* such that in these regions the parameters are perturbed by the interaction with the adjacent medium. Regions of unperturbed parameters are designated by D and D^* We write $V = D + S$ and $V^* = D^* + S^*$.

Equations of motion for the transition region $S + S^*$ are written in the form

$$-\omega^2\rho_S u_S + S\Phi u = q_S, \quad -\omega^2\rho_{S^*} u_{S^*} + S^*\Phi u = q_{S^*} \qquad (1.4)$$

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*) This particular characteristic is most convenient in the formulation of boundary value problems.

Here S and S^* are operators of multiplication by characteristic functions of boundary regions, $u_S = Su$, etc. Due to the presence of long-range interaction the terms $S\Phi u$ and $S^*\Phi u$ are responsible for the coupling of these equations among themselves and with equations for the unperturbed regions. Equation (1.4) represents the analog of coupling conditions in the local theory of a continuous medium. In particular, if the perturbed bonds are severed, these equations must yield the boundary conditions of force for each medium.

Turning now to the formulation of fundamental boundary value problems, we split the operator of elastic couplings into a sum of operators

$$\Psi = \Psi_V + \Psi_{V^*} + \Psi_{VV^*} \tag{1.5}$$

$$\Psi_V = V\Psi V, \qquad \Psi_{V^*} = V^*\Psi V^*, \qquad \Psi_{VV^*} = V\Psi V^* + V^*\Psi V$$

Here Ψ_V and Ψ_{V^*} characterize the interaction of points of the respective media. Ψ_{VV^*} corresponds to interaction between media (in the figure to each one of the operators corresponds its own type of coupling). Substituting (1.5) into (1.3) we find the corresponding decomposition of the operator

$$\Phi = \Phi_V + \Phi_{V^*} + \Phi_{VV^*} \tag{1.6}$$

Here for example (I is the identity operator)

$$\Phi_V = \psi_V \mathrm{I} - \Psi_V, \qquad \psi_V(x) = \int \Psi_V(x, x')\, dx' \tag{1.7}$$

If the media do not interact with each other, then $\Psi_{VV^*} = 0$ and consequently $\Phi_{VV^*} = 0$. In this case equations (1.1) are split into independent equations of both media. For the medium in region V we have

$$-\omega^2 \rho u_V + \Phi_V u_V = q_V \tag{1.8}$$

It is easy to see that Φ_V allows the following representation

$$\Phi_V = D\Phi + \Gamma, \qquad \Gamma = S\Phi_V \tag{1.9}$$

This permits us to write (1.8) in the form of the following equivalent system (*)

$$D\Phi_\omega u \equiv -\omega^2 \rho u_D + D\Phi u = q_D, \qquad \Gamma_\omega u \equiv -\omega^2 \rho u_S + \Gamma u = q_S \tag{1.10}$$

The first equation connects displacements with forces in region D. For $l \to 0$ it transforms into the equation of motion for the usual (local) theory of elasticity. The second equation relates displacements and forces in the boundary region S and can be obtained from conditions of coupling (1.4). For $l \to 0$ this equation transforms into the usual boundary conditions of force. This analogy permits us to call the formulated boundary value problem (1.10), the first fundamental problem of nonlocal theory of elastisity. The integral equation (1.8) is equivalent to this formulation.

A semi-bounded region was examined above. The generalization to a bounded region V is obvious.

We note that the static problem for the bounded region can be reduced to a Fredholm integral equation of the second kind with a symmetrical positive definite kernel if the assumption is made that all elastic couplings are stable, i.e. $\Psi(x, x') \geqslant 0$ which is natural for a mechanical system. This equation is obtained from (1.8) by standard substitu-

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*) In the notation of displacement of the region V it is convenient to omit the subscript V where the corresponding section is contained in the operators.

tion of variables and has the form

$$v(x) - \int_V K(x, x') v(x') dx' = q^\circ(x) \qquad (x \in V) \qquad (1.11)$$

$$v(x) = \psi_V^{1/2}(x) u(x), \quad q^\circ(x) = \psi_V^{-1/2}(x) q(x), \quad K(x, x') = [\psi_V(x) \psi_V(x')]^{-1/2} \Psi(x, x') \qquad (1.12)$$

A distinctive property of Eq. (1.11) is that the kernel $K(x, x') = 0$ for $|x - x'| > l$. Here the parameter l, as a rule, is small in comparison to the dimensions of the region.

Let us now turn to the second fundamental problem where in the boundary region the displacements, and not the forces, are given, i.e.

$$D\Phi_\omega u = q_D, \qquad u_S = h \qquad (1.13)$$

The function $h(x)$ is given on S. For this problem we can construct an equivalent integral equation

$$-\omega^2 \rho u_D + D\Phi u_D = f_D \qquad (f_D = q_D - D\Phi h) \qquad (1.14)$$

As with the first equation, in the static case this equation can be reduced to a Fredholm equation of the second kind with a symmetrical kernel.

We note especially the important case of the homogeneous medium. In this case $\Psi(x, x') = \Psi(x - x')$ and $\psi(x) = \psi_0$, $\rho(x) = \rho_0$ are constants. Equation (1.14) transforms into a Fredholm equation with a difference kernel

$$(-\omega^2 \rho_0 + \psi_0) u(x) - \int_D \Psi(x - x') u(x') dx' = f(x) \qquad (x \in D) \qquad (1.15)$$

The difference between the first and the second fundamental problem should be emphasized. For a homogeneous medium the first problem does not reduce to an integral equation with a difference kernel by virtue of essential inhomogeneity of elastic couplings in the boundary region. This, as will be seen later, leads to the situation where the solution of the first problem for the homogeneous medium in the nonlocal theory of elasticity is considerably more complicated than the solution of the second problem.

Developing further the analogy with the usual theory of elasticity, Green's formula is constructed in the nonlocal theory.

It follows from (1.7) that $\Phi_V = \Phi_V^+$, where Φ_V^+ is the conjugate operator. Taking into account (1.9), we obtain the operator identity

$$D\Phi + \Gamma = \Phi D + \Gamma^+$$

Both parts of the equation are applied to u and an inertial term is added in the form

$$\omega^2 \rho u_V = \omega^2 \rho u_D + \omega^2 \rho u_S$$

Taking into consideration (1.10), we find

$$\Phi_\omega u_D = q_D + \Gamma_\omega u - \Gamma_\omega^+ u_S \qquad (1.16)$$

Here $\Gamma_\omega u = q_S$ is the analog of density of the simple layer. $\Gamma_\omega^+ u_S$ is the analog of the double layer with density u_S. For $l \to 0$ they convert to the usual layers.

Let $G_\omega(x, x')$ denote the fundamental solution of Eq. (1.1). Applying operator G_ω to (1.16), we obtain Green's formula

$$u_D = G_\omega q_D + G_\omega q_S - G_\omega \Gamma_\omega^+ u_S \qquad (1.17)$$

From this formula we can find the representation of the solution of the first fundamental problem through Green's function

$$u_V(x) = \int_D G_\omega(x, x') q_D(x') dx' + \int_S G_\omega(x, x') q_S(x') dx' \qquad (x \in V)$$

if $G_\omega(x, x')$ is subjected to the following condition:

$$\int \Gamma_\omega(x, x'') G_\omega(x'', x') dx'' = \delta(x - x') \quad (x, x' \in S)$$

The solution of the second fundamental problem has the form

$$u_D(x) = \int_D G_\omega(x, x') q_D(x') dx' - \int_S dx' \int_D dx'' G_\omega(x, x'') \Gamma_\omega^+(x'', x') u_S(x') \quad (x \in D)$$

with the condition $G_\omega(x, x') = 0$, if x or x' belong to S.

Apparently Green's functions are the resolvents of the corresponding integral equations (1.8) and (1.14).

With the aid of (1.17) we can also formulate the analog to the mixed problem of the theory of elasticity.

In conclusion we emphasize that the obtained results are valid also for a discrete medium. In this case the integral equations for a bounded region are equivalent to a system of algebraic equations.

2. Let us turn to the examination of a homogeneous medium. We start with an investigation of the general solution of the equation of motion (1.1) for this case. This will permit us to examine the structure of solutions of various boundary value problems from a unified point of view. After Fourier transformation with respect to x (k-representation) Eq. (1.1) for the homogeneous medium takes the form

$$\Phi_\omega(k) u(k) \equiv [-\omega^2 \rho_0 + \Phi(k)] u(k) = q(k) \tag{2.1}$$

where $\Phi(k)$ is the Fourier transform of $\Phi(x)$. Taking into consideration (1.2) and (1.3), we have

$$\Phi(k) = 2 \int_0^l \Psi(x)(1 - \cos kx) dx \tag{2.2}$$

We can show [12] that functions of this form are analytically continued into the complex plane k as entire functions of the first order of growth and of the l-type. For an absolutely integrable $\Psi(x)$ the function $\Phi(k)$ is bounded on the real axis. It follows from real and even properties of $\Phi(k)$ that if $\Phi(k_1) = 0$, then $-k_1$, $\bar{k}_1$ and $-\bar{k}_1$ (the bar indicates complex conjugate values) will also be zeros of $\Phi(k)$.

Let us assume that all couplings are stable, i.e. $\Psi(x) \geqslant 0$. Then it is easy to see that $\Phi(k)$ does not have zeros on the real and imaginary axes with the exception of the double root $k = 0$. When the indicated conditions are fulfilled, an expansion with respect to roots [13] of the following form is appropriate

$$\Phi(k) = c_0 k^2 \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_n^2}\right) \qquad \left(c_0 = \int_0^l x^2 \Psi(x) dx\right) \tag{2.3}$$

Here k_n are roots of $\Phi(k)$ which are located in the upper half-plane and are renumbered in the order of increasing modulus. As will become clear later, the constant c_0 will be the modulus of elasticity of the medium in the zeroth long-wave approximation.

We note that for the discrete model, $\Phi(k)$ is a periodic function. The values of k are subject to the additional condition $|\mathrm{Re}\, k| \leqslant \pi / a$ [6, 14]. Consequently, the region of allowable values of k will be a complex cylinder. The number of roots k_n is equal to $2N$, where N is the number of interacting neighbors.

The general solution of the homogeneous equation corresponding to (2.1) can be obtained by superposition of waves of the form $\exp[ik(\omega)x]$, where $k(\omega)$ is found from the dispersion equation

$$\omega^2\rho_0 = \Phi(k) \tag{2.4}$$

Characteristically, the nonlocal theory exhibits for each ω the existence of an even number of, generally speaking, complex roots $k_n(\omega)$ of the dispersion equation (for the discrete model with limited long-range interaction their number, as was noted, is finite). For the unbounded medium the real $k_n(\omega)$ which correspond to undamped waves, are of fundamental interest. These waves are completely characterized by the dispersion curve $\omega = \omega(k)$, $\operatorname{Im} k = 0$. The group velocity $\omega'(k)$ of these waves depends on k (spatial dispersion). It is important to emphasize that the boundedness of $\Phi(k)$ leads to the appearance of a limit frequency $\omega = \omega_{\max}$ for undamped waves. In this manner, in contrast to generally accepted concepts [14], this effect exists not only in descrete media.

In the study of wave scattering on homogeneities and on boundaries of separation of homogeneous media and also for boundary value problems the complex $k_n(\omega)$ which correspond to damped waves have an essential significance along with the real values.

For fixed $\omega \neq 0$ we have in analogy to (2.3)

$$\Phi_\omega(k) = -\omega^2\rho_0 \prod_{n=0}^{\infty}\left(1 - \frac{k^2}{k_n^2(\omega)}\right) \quad \left(k_0^2(\omega) \to \frac{\rho_0\omega^2}{c_0} \text{ при } \omega \to 0\right) \tag{2.5}$$

The remaining $k_n(\omega)$ are renumbered in such a manner that $\operatorname{Im} k_n(\omega) \geqslant 0$.

The general solution of the inhomogeneous equation (2.1) in the x-representation has the form

$$u(x) = \int G_\omega(x - x')\, q(x')\, dx' + \sum_{n=0}^{\infty} [\alpha_n e^{ik_n(\omega)x} + \beta_n e^{-ik_n(\omega)x}] \tag{2.6}$$

Here the first term is the particular solution constructed with the aid of Green's function $G_\omega(x)$ of the unbounded medium, α_n and β_n are arbitrary constants.

It follows from (2.5) that $G_\omega(k) = \Phi_\omega^{-1}(k)$ is a meromorphic function. With some not very strong limitations on $\Phi_\omega(k)$ we can write the expansion of $G_\omega(k)$ into simple fractions in the following form:

$$G_\omega(k) = -\frac{1}{\omega^2\rho_0} + 2k^2 \sum_{n=0}^{\infty} \frac{1}{k_n(\omega)\,\Phi'[k_n(\omega)]\,[k^2 - k_n^2(\omega)]} \quad (\omega \neq 0) \tag{2.7}$$

Here the following evaluation is applicable (with the assumption that $\Phi'[k_n(\omega)] \neq 0$)

$$\left| \frac{1}{k_n(\omega)\,\Phi'(k_n(\omega))} \right| \leqslant A(\omega)\, e^{-l \operatorname{Im} k_n(\omega)}$$

In x-representation

$$G_\omega(x) = -\frac{1}{\omega^2\rho_0}\delta(x) - D_x^2 \sum_{n=0}^{\infty} \frac{i e^{ik_n(\omega)|x|}}{k_n^2(\omega)\Phi'[k_n(\omega)]} \quad \left(D_x = \frac{d}{dx}\right) \tag{2.8}$$

In the limiting case of $\omega \ll \omega_{\max}$, expression (2.7) assumes the form

$$G_\omega(k) = \frac{1}{c_0^2k^2 - \omega^2\rho_0} + g_0 + 2k^2 \sum_{n=1}^{\infty} \frac{1}{k_n\Phi'(k_n)(k^2 - k_n^2)} + O(\omega^2)$$

$$g_0 = \frac{1}{12\, c_0^2} \int_0^l x^4 \Psi(x)\, dx \tag{2.9}$$

In the static case (x-representation) we have

$$G(x) = \frac{|x|}{2c_0} + g_0\delta(x) - D_x^2 \sum_{n=1}^{\infty} \frac{i e^{ik_n|x|}}{k_n^2\Phi'(k_n)} \tag{2.10}$$

where the first term is the usual static Green's function.

If for some ω there are multiple roots (corresponding for example to extreme values of the dispersion curve), then among the solutions of the homogeneous equation there will be characteristic functions of the form $x^\mu \exp[ik_n(\omega)x]$ ($\mu = 0, 1, \ldots, m-1$, where m is the multiplicity of the root $k_n(\omega)$). The equations presented above change in an obvious manner.

Specific differences between the nonlocal and the classical theory which become apparent already in the one-dimensional case were pointed out above. Here we enumerate the basic ones: (a) a scale parameter exists; (b) the boundary is replaced by a boundary region; (c) new undamped and damped waves (*) appear (in statics damped characteristic functions appear); (d) a limit frequency exists for undamped waves; (e) the velocity of waves depends on their length.

It is natural to try to construct the simplest possible approximate models which quantitatively or qualitatively take correctly into account some of these effects. Let us examine from this point of view various approximate models and the regions of their applicability.

3. The simplest models can be obtained if $\Phi(k)$ is approximated by a polynomial $c_0k^2P_m(k^2)$, where $P_m(\lambda)$ is a real polynolial of λ of the mth degree. This corresponds to a replacement of the integral operator by a differential operator. The equations of motion take the form which is characteristic for phenomenological theories with higher derivatives (such as the couple-stress theory, multipolar theory, etc.)

$$\omega^2\rho_0 u(x) + c_0D_x^2 P_m(-D_x^2)u(x) = -q(x) \tag{3.1}$$

Two, in principle different, approaches are possible. In the final analysis the physical significance and the region of applicability of such models depend on these approaches. The assumption is usually accepted that the polynomial $c_0k^2P_m(k^2)$ represents a section of the series $\Phi(k)$ in the vicinity of $k = 0$. If $\Phi(k)$ is given or the behavior of the dispersion curve is known in the region of small k (long waves), then the first approximation can be obtained assuming $P_1(k^2) = 1 + l^2A_1k^2$, where A_1 is an appropriate nondimensional constant. This approximation allows to account correctly for the dispersion of long waves. This dispersion must be considered as weak. These approximations are called long-wave approximations, and the corresponding models are referred to as media with weak spatial dispersion. The scale parameter l must be considered as small, and the theory strictly speaking is not applicable to distances of the order of l, and it is even less applicable to distances less than l. If $P_1(k^2)$ is extrapolated into the region of large k (short waves), it is formally possible to obtain additional characteristic functions corresponding to roots of $P_1(k^2)$. However, these functions, generally speaking, will not have anything in common with the exact functions. Consequently, this approximation is not suitable for boundary value problems. We note that for any approximating polynomial which satisfies the condition of stability a limit frequency does not exist for undamped waves.

*) In the three-dimensional case new types of surface waves are also possible.

Another possible method for the approximation of $\Phi(k)$ consists in the construction of an interpolating polynomial in first roots. Taking into account peculiarities of distribution of roots of $\Phi(k)$, we find that for $\Psi(x) \geqslant 0$ the polynomial of the first approximation must have the form

$$\Phi_1(k) = c_0 k^2 \left(1 - \frac{k^2}{k_1^2}\right)\left(1 - \frac{k^2}{\bar{k}_1^2}\right) \tag{3.2}$$

The corresponding differential operator is of the sixth order.

Apparently $\Phi_1(k)$ cannot claim the same accuracy of approximation in the region of small k as $c_0 k^2 P_1(k^2)$. However, advantages of this model will be a qualitatively correct description of phenomena for which waves with a length of the order of l are essential, a preservation of the principal terms of the asymptotics of solutions, and the possibility of correct approximate formulation of boundary value problems.

For the adopted model with the assumption $\omega < \omega_{max}$ we have

$$\Phi_\omega(k) \approx -\omega^2\rho_0 \prod_{n=0}^{2}\left(1 - \frac{k^2}{k_n^2(\omega)}\right) \approx -\omega^2\rho_0 + \Phi_1(k) \tag{3.3}$$

Here the second representation is correct for not too high frequencies. When this condition is fulfilled, Green's function $G_\omega(k)$ in the approximation using first roots has the form

$$G_\omega(x) \approx \sum_{n=0}^{2} \frac{i e^{ik_n(\omega)|x|}}{\Phi_1'[k_n(\omega)]} \tag{3.4}$$

For the approximated static Green's function $G_1(x)$ we obtain

$$G_1(x) = \frac{|x|}{2c_0} + 2\,\mathrm{Re}\,\frac{i e^{ik_1|x|}}{\Phi_1'(k_1)} \tag{3.5}$$

If it is not required that $\Psi(x) \geqslant 0$, then roots $\Phi(k)$, generally speaking, may lie on the imaginary axis. In this case the approximated differential operator will be of the fourth order and the characteristic functions of the operator do not oscillate. The corresponding equations of motion can be viewed as the one-dimensional analog of equations of the couple-stress theory of elasticity.

Let us also mention other possible approximate models not connected with approximation by polynomials. In a number of case it may be of interest to prescribe an approximation of the dispersion curve over a wide range of waves with the aid of an appropriate function k (e.g. in the case of interpolation of experimental data). This model is good for the description of undamped waves, but it cannot be extended to the complex plane and it cannot be used in boundary value problems.

Models based on approximations of the following type can have a wider range of applicability

$$\Psi(x) = 3\,c_0 l^{-3} \quad (|x| \leqslant l)$$

$$\Psi(x) = 0 \quad (|x| > l)$$

$$\Psi(x) = (2/\pi)^{1/2} c_0 l^{-3} \exp(-x^2/2l^2)$$

These equations correspond to

$$\Phi(k) = 6\,c_0 l^{-2}\left(1 - \frac{\sin kl}{kl}\right) \qquad \Phi(k) = 2c_0 l^{-2}\left(1 - e^{-1/2(kl)^2}\right)$$

Such models describe, qualitatively, correctly all long-range effects.

4. As an illustration we shall examine the solutions of fundamental static problems in a homogeneous medium for a semi-bounded region D $(0 \leqslant x < \infty)$ and a bounded region $S(-l \leqslant x < 0)$.

Let us start with the simpler second fundamental peoblem

$$\int \Phi(x - x')u(x')dx' = q(x) \quad (x \in D), \quad u_S(x) = h(x) \quad (x \in S) \tag{4.1}$$

Assuming in (1.15) that $\omega = 0$ and selecting the units of measurement such that $\psi_0 = 1$, we obtain the equivalent integral equation in the form

$$u(x) - \int_0^\infty \Psi(x - x')\, u(x')\, dx' = f(x) \quad (0 \leqslant x < \infty)$$

An equation of this form was examined in [15] with the additional condition $\Phi(k) = 1 - \Psi(k) \neq 0$ $(-\infty < k < \infty)$ which is not fulfilled in the present case. Apparently the method in paper [15] allows the proper generalization (see also [16]). However, in order to improve clarity and also to obtain an effective approximation, the solution below will be constructed on the basis of characteristic functions of operator Φ.

We shall seek a solution, bounded for $x \to \infty$, in the form of a sum $u = u^* + v$, where $u^*(x)$ is a particular solution of Eq. (4.1) vanishing for $x \to \infty$, while $v(x)$ is the solution of the corresponding homogeneous equation and satisfies the boundary condition $v_S = h - u_S^*$. To obtain $u^*(x)$, we introduce the fundamental solution $G^*(x)$ of Eq. (4.1). This solution is connected with Green's function (2.10) by the relationship $G^*(x) = G(x) - x / 2c_0$. Then

$$u^*(x) = \int_D G^*(x - x')\, q_D(x')\, dx' \tag{4.2}$$

Apparently, $v(x)$ can be represented by superposition of characteristic functions of the operator Φ. The functions are not increasing for $x \to \infty$

$$v(x) = \sum_{n=0}^{\infty} v^n e_n(x), \qquad e_n(x) = e^{ik_n x} \qquad (k_0 = 0, \ \operatorname{Im} k_n > 0) \tag{4.3}$$

The coefficients of expansion are found if a system of functions $e^m(x)$ is constructed so that these functions are concentrated on S and form a reciprocal basis

$$(e^m, e_n) \equiv \int e^m(x)\, e^{ik_n x}\, dx = e^m(k_n) = \delta_n{}^m \qquad (m, n = 0, 1, \ldots) \tag{4.4}$$

($\delta_n{}^m$ is Kronecker's symbol)

For this purpose we take advantage of the following representation [15]

$$\Phi(k) = \Phi_+(k)\,\Phi_-(k) \qquad (\Phi_-(k) = \Phi_+(-k)) \tag{4.5}$$

Here $\Phi_\pm$ are entire functions of the first order of growth and of the l-type. These functions also do not have roots within the upper (lower) half-plane. Inverse Fourier transforms of these functions are concentrated in intervals $[0, l]$ and $[-l, 0]$, respectively. It is not difficult to show that in the lower half-plane $\Phi_-(k)$ and $\Phi(k)$ are connected through the following relationship

$$\ln \Phi_-(k) = \frac{1}{2\pi i} \int_L \frac{\ln \Phi(k')}{k' - k}\, dk' \qquad (\operatorname{Im} k > 0)$$

The real axis serves as the contour of integration with circumvention of the origin of coordinates from below. It is easy to verify that when $\Phi_-(k)$ is given the functions with reciprocal basis are found from the following equations

$$e^m(k) = \frac{\Phi_-(k)}{\Phi_-{}'(k_m)\,(k - k_m)} \tag{4.6}$$

It is possible to show that $e^m(x)$ are concentrated in S and have the form

$$e^m(x) = \frac{i}{\Phi_-'(k_m)} \int_{-\infty}^{0} \Phi_-(x-x')\, e^{-ik_m x'}\, dx' \tag{4.7}$$

From (4.3) and (4.4) we obtain

$$v^n = (e^n,\ v_S) \quad (n = 0,1,\ldots) \tag{4.8}$$

By adding (4.2) and (4.3) we find

$$u(x) = \int_D G^*(x-x')\, q_D(x')\, dx' + \sum_{n=0}^{\infty} v^n e_n(x) \qquad (x \in D) \tag{4.9}$$

The solution can also be represented in the form

$$u(x) = \int_D G(x,\ x')\, f(x')\, dx' \qquad (x \in D) \tag{4.10}$$

where $G(x, x')$ is Green's function for the given problem. It is easy to check that $G(x, x')$ allows the following representation:

$$G(x,\ x') = G^*(x-x') + \int_S E(x,\ x'')\, G(x'',\ x')\, dx'', \qquad E(x,\ x') = \sum_{n=0}^{\infty} e_n(x)\, e^n(x') \tag{4.11}$$

In the approximation which utilizes first roots, the solution is determined by characteristic functions of the operator (3.2). This solution can be constructed by the same method as the exact solution if the corresponding approximate expressions for G^* and for the first functions of the reciprocal basis are obtained. From (3.5) we find

$$G^*(x) \approx -c_0^{-1} x \eta(-x) + 2\,\mathrm{Re}\, \frac{ie^{ik_1|x|}}{\Phi_1'(k_1)} \tag{4.12}$$

where $\eta(x)$ is the function of unit jump.

The roots $\Phi_-(k)$ are located on the upper half-plane. Therefore, from (3.2) and (4.5) it follows directly that $\Phi_-(k) \approx \sqrt{c_0}k(1 - k / k_1)\ (1 + k / \bar{k}_1)$

and consequently we have according to (4.6)

$$e^0(k) \approx \left(1 - \frac{k}{k_1}\right)\left(1 + \frac{k}{\bar{k}_1}\right), \qquad e^1(k) \approx \frac{k(k+\bar{k}_1)}{k_1(k_1+\bar{k}_1)} \tag{4.13}$$

Taking into account these relationships and utilizing Eqs. (4.8) for coefficients v^n it is easy to ascertain that the boundary conditions are expressed through derivatives $u(x)$ at the point $x = 0$

$$\left(1 - \frac{iD_x}{k_1}\right)\left(1 + \frac{iD_x}{\bar{k}_1}\right) u(0) = u^0 \equiv (e^0, h)$$

$$\frac{iD_x(iD_x + k_1)}{k_1(k_1+\bar{k}_1)}\, u(0) = u^1 \equiv (e^1, h) \tag{4.14}$$

the second condition is equivalent to two real conditions here.

Now let us turn to the force problem (1.10) which in the given case assumes the form

$$\int \Phi(x-x')\, u(x')\, dx' = q_D(x) \qquad (x \in D), \qquad \int \Gamma(x, x')\, u(x')\, dx' = q_S(x) \qquad (x' \in S) \tag{4.15}$$

Just as previously, we shall be looking for a solution in the form (4.9). However, now it is appropriate to write $e_0(x) = x$, since $u(x)$ is determined with accuracy to a constant. Substituting u into the boundary conditions, we obtain

$$\sum_{n=0}^{\infty} v^n \Gamma e_n(x) = Q(x), \qquad Q(x) = q_S(x) - \Gamma u^*(x)$$

The construction of the basis corresponding to Γe_n for finding coefficients v^n represents a significantly more difficult problem than the construction of e^n. This is a consequence of inhomogeneity of couplings in S. In the general case the problem is reduced to the solution of an infinite system of equations

$$\sum_{n=0}^{\infty} \Gamma_{mn} v^n = Q_m, \qquad \Gamma_{mn} = (e_m, \Gamma e_n), \qquad Q_m = (e_m, Q) \tag{4.16}$$

However, the problem simplifies considerably in the approximation using first roots. In this case we have for the basis e^n (cf. (4.13))

$$e^0(k) \approx ik\left(1 - \frac{k}{k_1}\right)\left(1 + \frac{k}{\bar{k}_1}\right), \qquad e^1(k) \approx \frac{k^2}{k_1^2}\,\frac{k + \bar{k}_1}{k_1 + \bar{k}_1}, \qquad e^2(k) = \bar{e}^1(-k) \tag{4.17}$$

The boundary conditions assume the form ($m = 0.1$)

$$\Gamma_m(D_x) u(0) = q_m, \quad \Gamma_m = \sum_{n=0}^{2} \Gamma_{mn} e^n(iD_x), \quad q_m = (e_m, q_S) \tag{4.18}$$

Here $e^n(iD_x)$ are differential operators corresponding to (4.17).

We note that in contrast to couple-stress theories of elasticity, here the connection between approximate and exact boundary conditions is clear. In particular, a method is indicated for the computation of moments q_m from forces q_S given on the boundary.

5. Let us examine the simplest generalization to the case of a three-dimensional medium with central interaction. In such a medium the forces which arise when the distances between the points $r(x^\alpha)$ and $r'(x'^\alpha)$ are changed, are proportional to this change and are directed along the line connecting the points. It is possible to show that such a medium is described by the operator of elastic couplings $\Psi^{\alpha\beta}$ with the kernel

$$\Psi^{\alpha\beta}(\mathbf{r}, \mathbf{r}') = \frac{(x^\alpha - x'^\alpha)(x^\beta - x'^\beta)}{|\mathbf{r} - \mathbf{r}'|^2}\Psi(\mathbf{r}, \mathbf{r}'), \qquad \Psi(\mathbf{r}, \mathbf{r}') = \Psi(\mathbf{r}', \mathbf{r}) \tag{5.1}$$

In this connection it is assumed that $\Psi(\mathbf{r}, \mathbf{r}') = 0$, if $|\mathbf{r} - \mathbf{r}'| > l$.

Repeating almost verbatim the arguments presented for the one-dimensional case, we can introduce three-dimensional operators $\Phi^{\alpha\beta}_{(\omega)}$, $\Gamma^{\alpha\beta}_{(\omega)}$, etc., to formulate the fundamental boundary value problems, and to write Green's formula. In particular, with obvious notation, Green's formula for the three-dimensional case has the form

$$u_\alpha = G_{\alpha\beta}{}^{(\omega)} q_D{}^\beta + G_{\alpha\beta}{}^{(\omega)} q_S{}^\beta - G_{\alpha\beta}{}^{(\omega)} \Gamma_{(\omega)\lambda}{}^{+\beta} u_S{}^\lambda \tag{5.2}$$

In connection with the fact that the radius of forces of long-range interaction is usually small compared to other characteristic dimensions, the boundary value problems for the half-space (the half-plane) are of fundamental interest in the nonlocal theory of elasticity. The solution of these problems can be obtained by a method which is analogous to the method examined above for a half-axis if Green's function is known for the unbounded medium. The latter can be easily constructed for a homogeneous isotropic medium. In this case

$$\Psi(\mathbf{r}, \mathbf{r}') = \Psi(\mathbf{r} - \mathbf{r}'), \qquad \Psi(\mathbf{r}) = \Psi(r) \quad (r = |\mathbf{r}|)$$

$$\Phi^{\alpha\beta}(\mathbf{r}) = \psi_0{}^{\alpha\beta}\delta(\mathbf{r}) - \Psi^{\alpha\beta}(\mathbf{r}), \qquad \psi_0{}^{\alpha\beta} = \int \Psi^{\alpha\beta}(\mathbf{r})\, d\mathbf{r} \tag{5.3}$$

Let us break up the energy operator $\Phi^{\alpha\beta}$ into a longitudinal $\Phi^{\alpha\beta}_{(l)}$ and a transverse $\Phi^{\alpha\beta}_{(t)}$ component. We can show that the following representation ($k = |\mathbf{k}|$) is valid for the Fourier transforms of the corresponding kernels

$$\Phi_{(l)}{}^{\alpha\beta}(\mathbf{k}) = \frac{k^\alpha k^\beta}{k^2}\Phi_{(l)}(k), \qquad \Phi_{(t)}{}^{\alpha\beta}(\mathbf{k}) = \left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right)\Phi_{(t)}(k)$$
$$\Phi_{(l,t)}(k) = 4\pi\int_0^l r^2\Psi_{(l,t)}(r)\left(1 - \frac{\sin kr}{kr}\right)dr \tag{5.4}$$

Here $\Psi_{(l)}$, $\Psi_{(t)}$ and Ψ are connected by relationships

$$\Psi(r) = \Psi_{(l)}(r) + 2\Psi_{(t)}(r), \qquad \Psi_{(t)}(r) = \int_r^l r^{-1}\Psi(r)\,dr \tag{5.5}$$

In accordance with (5.4) we can write the separation of static Green's function into a longitudinal and a transverse component

$$G_{\alpha\beta}(\mathbf{k}) = \frac{k_\alpha k_\beta}{k^2}G_{(l)}(k) + \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)G_{(t)}(k), \qquad G_{(l,t)}(k) = \Phi_{(l,k)}{}^{-1}(k) \tag{5.6}$$

For the condition $\Psi(r) \geqslant 0$ the properties of entire functions $\Phi_{(l,t)}(k)$ and in particular the distribution of roots are analogous to properties of $\Phi(k)$ of the one-dimensional medium. This makes it possible to separate $G_{(l,t)}(k)$ into partial fractions. For example

$$G_{(l)}(k) = \frac{1}{c_{(l)}k^2} + g_{(l)} + 2k^2\sum_{n=1}^{\infty}\frac{1}{k_n\Phi_{(l)}'(k_n)(k^2 - k_n{}^2)} \qquad (\operatorname{Im} k_n > 0)$$
$$c_{(l)} = \frac{2\pi}{3}\int_0^l r^4\Psi_{(l)}(r)\,dr, \qquad g_{(l)} = \frac{\pi}{30c_{(l)}{}^2}\int_0^l r^6\Psi_{(l)}(r)\,dr \tag{5.7}$$

In the r-representation we obtain (Δ is the Laplace operator)

$$G_{(l)}(r) = \frac{1}{4\pi}\left[\frac{1}{c_{(l)}r} + g_{(l)}\frac{\delta(r)}{r^2} - 2\Delta\sum_{n=1}^{\infty}\frac{1}{k_n\Phi'(k_n)}\frac{e^{ik_nr}}{r}\right] \tag{5.8}$$

For the approximation with respect to first roots we find in analogy to (3.5)

$$G_{(l)}(r) \approx \frac{1}{4\pi c_{(l)}r}\left[1 + 4c_{(l)}\operatorname{Re}\left(\frac{k_1e^{ik_1r}}{\Phi_{(l)}'(k_1)}\right)\right] \tag{5.9}$$

For $\operatorname{Re}k_1 = 0$ the expression which was obtained earlier in [5] follows from these equations.

In conclusion we note the characteristic boundary effect for the three-dimensional medium. The existence of new types of longitudinal and transverse waves must lead to the appearance of new surface waves. In contrast to Rayleigh waves, these waves decay for large lengths in a layer of the order of the parameter for long-range interaction.

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ON THE FOUNDATIONS OF A THEORY OF EQUILIBRIUM CRACKS IN ELASTIC SOLIDS

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Propositions expressed in [9] receive further development herein, are refined and systematized. Proceeding from the law of interaction between atoms, a conception is proposed which considers cracks in elastic solids as nontrivial modes of equilibrium deformation. Crack formation is treated as the loss of stability (in the large) of trivial equilibrium modes. The formulation of the brittle fracture criterion in the neighborhood of the end of the crack is refined. The carrying capacity of a solid having an equilibrium crack is estimated approximately in an example of the A. Griffith problem.

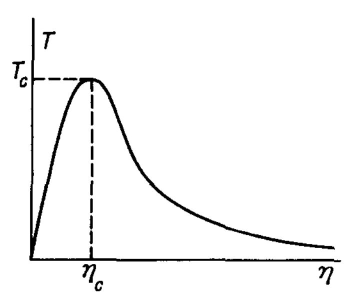

Fig. 1

1. Let $T = D\sigma$ (where $D = 2a$ is the atomic diameter) be the force of interaction between two